THE CELLULARIZATION PRINCIPLE FOR QUILLEN ADJUNCTIONS

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Abstract. The Cellularization Principle states that under rather weak conditions, a Quillen adjunction of stable model categories induces a Quillen equivalence on cellularizations provided there is a derived equivalence on cells. We give a proof together with a range of examples.

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1. Introduction

The purpose of this paper is to publicize a useful general principle when comparing model categories: whenever one has a Quillen adjunction

$$F : \mathbb{M} \rightleftarrows \mathbb{N} : U$$

comparing two stable model categories, we obtain another Quillen adjunction by cellularizing the two model categories with respect to corresponding objects and a Quillen *equivalence* provided the derived unit or counit is an equivalence on cells. In this case, the cellularization of the adjunction induces a homotopy category level equivalence between the respective localizing subcategories. The hypotheses are mild, and the statement may appear like a tautology. The Cellularization Principle can be directly compared to another extremely powerful formality, that a natural transformation of cohomology theories that is an isomorphism on spheres is an equivalence.

This result was first proved in an appendix of the original versions of [4], but the range of cases where the conclusion is useful led us to present the result separately from that particular application.

The paper is layed out as follows: in Section 2 we give the statement and proof of the Cellularization Principle, and the following sections give a selection of examples.

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2. Cellularization of model categories

Throughout the paper we need to consider models for categories of cellular objects, thought of as built from a set of basic cells using coproducts and cofibre sequences. These models are usually obtained by the process of cellularization (sometimes known as right localization or colocalization) of model categories, with the cellular objects appearing as the cofibrant objects. Because it is fundamental to our work, we recall some of the basic definitions from [7].

Definition 2.1. [7, 3.1.8] Let $\mathbb{M}$ be a model category and $\mathcal{K}$ be a set of objects in $\mathbb{M}$. A map $f : X \to Y$ is a *$\mathcal{K}$-cellular equivalence* if for every element A in $\mathcal{K}$ the induced map of homotopy function complexes [7, 17.4.2] $f_* : \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$ is a weak equivalence. An object W is *$\mathcal{K}$-cellular* if W is cofibrant in $\mathbb{M}$ and $f_* : \mathrm{map}(W, X) \to \mathrm{map}(W, Y)$ is a weak equivalence for any $\mathcal{K}$-cellular equivalence f.

One can cellularize a right proper model category under very mild finiteness hypotheses. To avoid confusion due to the dual use of the word "cellular" we recall that a *cellular* model category is a cofibrantly generated model category with smallness conditions on its generating cofibrations and acyclic cofibrations [7, 12.1.1].

Proposition 2.2. [7, 5.1.1] *Let $\mathbb{M}$ be a right proper, cellular model category and let $\mathcal{K}$ be a set of objects in $\mathbb{M}$. The $\mathcal{K}$-cellularized model category $\mathcal{K}$-cell-$\mathbb{M}$ exists and has weak equivalences the $\mathcal{K}$-cellular equivalences, fibrations the $\mathbb{M}$-fibrations, and cofibrations the maps with the left lifting property with respect to the trivial fibrations. The cofibrant objects are the $\mathcal{K}$-cellular objects.*

Remark 2.3. Since the $\mathcal{K}$-cellular equivalences are defined using homotopy function complexes, the $\mathcal{K}$-cellularized model category $\mathcal{K}$-cell-$\mathbb{M}$ depends only on the homotopy type of the objects in $\mathcal{K}$.

It is useful to have the following further characterization of the cofibrant objects.

Proposition 2.4. [7, 5.1.5] *If $\mathcal{K}$ is a set of cofibrant objects in $\mathbb{M}$, then the class of $\mathcal{K}$-cellular objects agrees with the smallest class of cofibrant ojects in $\mathbb{M}$ that contains $\mathcal{K}$ and is closed under homotopy colimits and weak equivalences.*

Throughout this paper we consider stable cellularizations of stable model categories. Say that a set $\mathcal{K}$ is *stable* if for any $A \in \mathcal{K}$ all of its suspensions (and desuspensions) are also in $\mathcal{K}$ up to weak equivalence. That is, since the cellularization only depends on the homotopy type of elements in $\mathcal{K}$, if $A \in \mathcal{K}$, then for all $i \in \mathbb{Z}$ there are objects $B_i \in \mathcal{K}$ with $B_i \simeq \Sigma^i A$. In this case, for $\mathbb{M}$ a stable model category and $\mathcal{K}$ a stable set of objects, $\mathcal{K}$-cell-$\mathbb{M}$ is again a stable model category; see [1, 4.6]. In this case, one can use homotopy classes of maps instead of the homotopy function complexes in Definition 2.1. That is, a map $f : X \to Y$ is a $\mathcal{K}$-cellular equivalence if and only if for every element A in $\mathcal{K}$ the induced map $[A, X]_* \to [A, Y]_*$ is an isomorphism; see [1, 4.4].

Proposition 2.5. *If $\mathbb{M}$ is a right proper, stable, cellular model category and $\mathcal{K}$ is stable, then $\mathcal{K}$ detects trivial objects. That is, in $\mathrm{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$, an object X is trivial if and only if for each element A in $\mathcal{K}$, $[A, X]_* = 0$.*

Proof: By [8, 7.3.1], the set of cofibres of the generating cofibrations detects trivial objects. In this stable situation, a set of generating cofibrations is produced in [1, 4.9]. Since $\mathcal{K}$ is stable, the cofibres of these maps are either contractible or are weakly equivalent to objects in $\mathcal{K}$ again. □

If in addition $\mathcal{K}$ consists of objects which are small in the homotopy category (referred to as *small* from now on), then by [10, 2.2.1] we have the following.

Corollary 2.6. *If* $\mathbb{M}$ *is a right proper, stable, cellular model category and* $\mathcal{K}$ *is a stable set of small objects, then* $\mathcal{K}$ *is a set of generators of* $\mathrm{Ho}(\mathcal{K}\text{-}cell\text{-}\mathbb{M})$. *That is, the only localizing subcategory containing* $\mathcal{K}$ *is* $\mathrm{Ho}(\mathcal{K}\text{-}cell\text{-}\mathbb{M})$ *itself.*

We next need to show that appropriate cellularizations of these model categories preserve Quillen adjunctions and induce Quillen equivalences. We assume that the adjoint Quillen functors are pointed; that is, that each functor preserves the zero object.

Proposition 2.7. The Cellularization Principle. *Let* $\mathbb{M}$ *and* $\mathbb{N}$ *be right proper, stable, cellular model categories with* $F : \mathbb{M} \to \mathbb{N}$ *a pointed left Quillen functor with pointed right adjoint* U. *Let* Q *be a cofibrant replacement functor in* $\mathbb{M}$ *and* R *a fibrant replacement functor in* $\mathbb{N}$.

(1) *Let* $\mathcal{K} = \{A_\alpha\}$ *be a set of objects in* $\mathbb{M}$ *with* $FQ\mathcal{K} = \{FQA_\alpha\}$ *the corresponding set in* $\mathbb{N}$. *Then* F *and* U *induce a Quillen adjunction between the* $\mathcal{K}$*-cellularization of* $\mathbb{M}$ *and the* $FQ\mathcal{K}$*-cellularization of* $\mathbb{N}$.

$$F : \mathcal{K}\text{-}cell\text{-}\mathbb{M} \rightleftarrows FQ\mathcal{K}\text{-}cell\text{-}\mathbb{N} : U$$

(2) *If* $\mathcal{K} = \{A_\alpha\}$ *is a stable set of small objects in* $\mathbb{M}$ *such that* FQA *is small in* $\mathbb{N}$ *and* $QA \to URFQA$ *is a weak equivalence in* $\mathbb{M}$ *for each* A *in* $\mathcal{K}$, *then* F *and* U *induce a Quillen equivalence.*

$$\mathcal{K}\text{-}cell\text{-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-}cell\text{-}\mathbb{N}$$

(3) *Let* $\mathcal{L} = \{B_\beta\}$ *be a stable set of small objects in* $\mathbb{N}$, *with* $UR\mathcal{L} = \{URB_\beta\}$ *the corresponding set of objects in* $\mathbb{N}$. *If for each* B *in* $\mathcal{L}$ *the map* $FQURB \to RB$ *is a weak equivalence in* $\mathbb{N}$ *and* URB *is small in* $\mathbb{M}$, *then* F *and* U *induce a Quillen equivalence between the* $\mathcal{L}$*-cellularization of* $\mathbb{N}$ *and the* $UR\mathcal{L}$*-cellularization of* $\mathbb{M}$.

$$UR\mathcal{L}\text{-}cell\text{-}\mathbb{M} \simeq_Q \mathcal{L}\text{-}cell\text{-}\mathbb{N}$$

Proof: Using the equivalences in [7, 3.1.6], the criterion in [7, 3.3.18(2)] (see also [9, 2.2]) for showing that F and U induce a Quillen adjoint pair on the cellularized model categories in (1) is equivalent to requiring that U takes $FQ\mathcal{K}$-cellular equivalences between fibrant objects to $\mathcal{K}$-cellular equivalences. Any Quillen adjunction induces a weak equivalence $\mathrm{map}(A, URX) \simeq \mathrm{map}(FQA, X)$ of the homotopy function complexes, see for example [7, 17.4.15]. So a map $f : X \to Y$ induces a weak equivalence $f_* : \mathrm{map}(FQA, X) \to \mathrm{map}(FQA, Y)$ if and only if $Uf_* : \mathrm{map}(A, URX) \to \mathrm{map}(A, URY)$ is a weak equivalence. Thus in (1), U preserves (and reflects) the cellular equivalences between fibrant objects. Hence, U induces a Quillen adjunction on the cellularized model categories.

Similarly, $Uf_* : \mathrm{map}(URB, URX) \to \mathrm{map}(URB, URY)$ is a weak equivalence if and only if $f_* : \mathrm{map}(FQURB, X) \to \mathrm{map}(FQURB, Y)$ is. Given the hypothesis in (3) that

$FQURB \to RB$ is a weak equivalence, it follows that Uf_* is a weak equivalence if and only if $f_* : \mathrm{map}(B, X) \to \mathrm{map}(B, Y)$ is. Thus, it follows in (3) that U preserves (and reflects) the cellular equivalences between fibrant objects. Hence, U induces a Quillen adjunction on the cellularized model categories. Note that the stability of $\mathbb{M}$, $\mathbb{N}$, $\mathcal{K}$ and $\mathcal{L}$ was not necessary for establishing the Quillen adjunction in (1) or (3).

Next we establish the Quillen equivalence in (2); the arguments are very similar for (3). Since $\mathbb{M}$ and $\mathcal{K}$ are stable, $\mathcal{K}$-cell-$\mathbb{M}$ is a stable model category by [1, 4.6]. Since left Quillen functors preserve homotopy cofibre sequences and F is pointed, $FQ\mathcal{K}$, and hence also $FQ\mathcal{K}$-cell-$\mathbb{N}$, are stable. The Quillen adjunction in (1) induces a derived adjunction on the triangulated homotopy categories; we show that this is actually a derived equivalence. Both derived functors are exact (since the left adjoint commutes with suspension and cofibre sequences and the right adjoint commutes with loops and fibre sequences). As a left adjoint, F also preserves coproducts. We next show that the right adjoint preserves coproducts as well.

Since $\mathcal{K} = \{A_\alpha\}$ detects $\mathcal{K}$-cellular equivalences, to show that U preserves coproducts it suffices to show that for each $A_\alpha \in \mathcal{K}$ and any family $\{X_i\}$ of objects in $\mathbb{N}$ the natural map

$$[A_\alpha, \coprod UX_i] \to [A_\alpha, U(\coprod X_i)]$$

is an isomorphism. Using the adjunction and the fact that each A_α is small, the source can be rewritten as

$$[A_\alpha, \coprod UX_i] \cong \oplus_i [A_\alpha, UX_i] \cong \oplus_i [FQA_\alpha, X_i].$$

Similarly, using the adjunction, the target is isomorphic to $[FQA_\alpha, \coprod X_i]$. Since FQA_α is assumed to be small, the source and target are isomorphic and this shows that U commutes with coproducts.

Consider the full subcategories of objects M in $\mathrm{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$ and N in $\mathrm{Ho}(FQ\mathcal{K}\text{-cell-}\mathbb{N})$ such that the unit $QM \to URFQM$ or counit $FQURN \to RN$ of the adjunctions are equivalences. Since both derived functors are exact and preserve coproducts, these are localizing subcategories. Since for each A in $\mathcal{K}$ the unit is an equivalence and $\mathcal{K}$ is a set of generators by Corollary 2.6, the unit is an equivalence on all of $\mathrm{Ho}(\mathcal{K}\text{-cell-}\mathbb{M})$. It follows that the counit is also an equivalence for each object $N = FQA$ in $FQ\mathcal{K}$. Since $FQ\mathcal{K}$ is a set of generators for $\mathrm{Ho}(FQ\mathcal{K}\text{-cell-}\mathbb{N})$, the counit is also always an equivalence. Statement (2) follows. $\square$

Note that if F and U form a Quillen equivalence on the original categories, then the conditions in Proposition 2.7 parts (2) and (3) are automatically satisfied. Thus, they also induce Quillen equivalences on the cellularizations.

Corollary 2.8. *Let $\mathbb{M}$ and $\mathbb{N}$ be right proper, stable cellular model categories with $F : \mathbb{M} \to \mathbb{N}$ a pointed Quillen equivalence with pointed right adjoint U. Let Q be a cofibrant replacement functor in $\mathbb{M}$ and R a fibrant replacement functor in $\mathbb{N}$.*

(1) *Let $\mathcal{K} = \{A_\alpha\}$ be a stable set of small objects in $\mathbb{M}$, with $FQ\mathcal{K} = \{FQA_\alpha\}$ the corresponding set of objects in $\mathbb{N}$. Then F and U induce a Quillen equivalence between the $\mathcal{K}$-cellularization of $\mathbb{M}$ and the $FQ\mathcal{K}$-cellularization of $\mathbb{N}$:*

$$\mathcal{K}\text{-}cell\text{-}\mathbb{M} \simeq_Q FQ\mathcal{K}\text{-}cell\text{-}\mathbb{N}$$

(2) *Let $\mathcal{L} = \{B_\beta\}$ be a set of small objects in $\mathbb{N}$, with $UR\mathcal{L} = \{URB_\beta\}$ the corresponding set of objects in $\mathbb{N}$. Then F and U induce a Quillen equivalence between the $\mathcal{L}$-cellularization of $\mathbb{N}$ and the $UR\mathcal{L}$-cellularization of $\mathbb{M}$:*

$$UR\mathcal{L}\text{-}cell\text{-}\mathbb{M} \simeq_Q \mathcal{L}\text{-}cell\text{-}\mathbb{N}$$

In [9, 2.3] Hovey gives criteria for when localizations preserve Quillen equivalences. Since cellularization is dual to localization, a generalization of this corollary without stability or smallness hypotheses follows from the dual of Hovey's statement.

3. Smashing localizations

We suppose given a map $\theta : S \longrightarrow R$ of commutative ring spectra (or DGAs). This gives the extension and restriction of scalars Quillen adjunction

$$\theta_* : S\text{-mod} \rightleftarrows R\text{-mod} : \theta^* \ ,$$

where $\theta_* N = R \wedge_S N$. We apply Proposition 2.7 with $\mathbb{M} = S$-mod and $\mathbb{N} = R$-mod. The category of R-modules is generated by the R-module R, and we use that as the generating cell. The following immediate corollary of the Cellularization Principle gives interesting cases in which the resulting Quillen adjunction is a Quillen equivalence.

Corollary 3.1. *If $R \wedge_S R \xrightarrow{\simeq} R$ is an equivalence, then we have a Quillen equivalence*

$$R\text{-}cell\text{-}S\text{-}modules \simeq_Q R\text{-}modules. \quad \square$$

The first example would be if S and R are conventional commutative rings or DGAs and $R = \mathcal{E}^{-1}S$ for some multiplicatively closed set $\mathcal{E}$. The condition is satisfied since $\mathcal{E}^{-1}S \wedge_S \mathcal{E}^{-1}S \cong \mathcal{E}^{-1}S$ and we find

$$\mathcal{E}^{-1}S\text{-cell-}S\text{-modules} \simeq_Q \mathcal{E}^{-1}S\text{-modules}.$$

More generally any smashing localization of module spectra behaves this way. We first suppose given a smashing localization L of the category of S-module spectra. The unit map gives a ring map $S \longrightarrow LS$ [3, VIII.2.2], so that we may apply the above discussion with $R = LS$. The smashing condition states that we have an equivalence $LN \simeq LS \wedge_S N$ for any S-module N.

Proposition 3.2. *Given a ring spectrum S and a smashing localization L on the category of S-modules, we have Quillen equivalences*

$$LS\text{-}cell\text{-}S\text{-}modules \simeq_Q L(S\text{-}modules) \simeq_Q LS\text{-}modules.$$

Proof: The smashing condition applies in particular to the generator $N = LS$, and hence the Cellularization Principle gives

$$LS\text{-cell-}S\text{-modules} \simeq_Q LS\text{-modules}.$$

We may also apply the Cellularization Principle to the localization functor, taking $\mathbb{N} = L(S\text{-mod})$. The smashing condition means that $L(S$-modules) is generated by the single object LS, and the universal property together with the smallness of S shows that LS is small. Accordingly, the Cellularization Principle shows

$$LS\text{-cell-}S\text{-modules} \simeq_Q \mathrm{L}(S\text{-modules}).$$

See also [3, VIII.3.2] and [10, 3.2(iii)]. $\square$

Perhaps the best known example in the category of spectra is when $S = \mathbb{S}$ is the sphere spectrum and $R = E_n$ is the nth p-local Morava E-theory, so that

$$L_n\mathbb{S}\text{-cell-}\mathbb{S}\text{-modules} \simeq L_n(\mathbb{S}\text{-modules}) \simeq (L_n\mathbb{S})\text{-modules}.$$

4. Isotropic equivalences of ring G-spectra

We suppose given a map $\theta : S \longrightarrow R$ of ring spectra. This gives the extension and restriction of scalars Quillen adjunction

$$\theta_* : S\text{-mod} \rightleftarrows R\text{-mod} : \theta^* .$$

We apply Propostion 2.7 with $\mathbb{M} = S$-mod and $\mathbb{N} = R$-mod. If the ring spectra are non-equivariant, then S generates S-modules and the Cellularization Principle shows we have an equivalence if θ is a weak equivalence of S-modules.

If the rings are in the category of G-spectra we get a somewhat more interesting example. The category of S-modules is generated by the extended objects $G/H_+ \wedge S$ as H runs through closed subgroups of G and the unit is the comparison $G/H_+ \wedge \theta$. We say θ is an $\mathcal{F}$-equivalence if $G/H_+ \wedge \theta$ is an equivalence for all H in a family $\mathcal{F}$. Define $\mathcal{F}$-cellularization of S-mod to be cellularization with respect to the set of all suspensions and desuspensions of objects $G/H_+ \wedge S$ for H in a family $\mathcal{F}$. Then the Cellularization Principle gives an equivalence

$$\mathcal{F}\text{-cell-}S\text{-module-}G\text{-spectra} \simeq \mathcal{F}\text{-cell-}R\text{-module-}G\text{-spectra}.$$

5. Torsion modules

Let R be a conventional commutative Noetherian ring and I an ideal. We apply Proposition 2.7, with $\mathbb{N}$ the category of differential graded R-modules and $\mathbb{M}$ the category of differential graded I-power torsion modules. There is an adjunction

$$i : I\text{-power-torsion-}R\text{-modules} \rightleftarrows R\text{-modules} : \Gamma_I$$

with left adjoint the inclusion and the right adjoint Γ_I defined by

$$\Gamma_I(M) = \{m \in M \mid I^N m = 0 \text{ for } N >> 0\}.$$

Both of these categories support injective model structures by [8, 2.3.13], with cofibrations the monomorphisms and weak equivalences the quasi-isomorphisms. For torsion-modules, one must modify the proof a bit; to construct products and inverse limits one forms them in the category of all R-modules and then applies the right adjoint Γ_I; see also [5, 8.6]. With these structures the above adjunction is a Quillen adjunction.

We now consider the Cellularization Principle with $\mathbb{M} = I$-power-torsion-R-modules and $\mathbb{N} = R$-modules. It is proved in [2], that the category of I-power torsion modules is generated by R/I. The fibrant replacement $f(iR/I)$ of $i(R/I)$ may be taken to be a resolution by injective I-power torsion modules, and hence Γ_I has no effect and the counit is an equivalence. We conclude

$$I\text{-power-torsion-}R\text{-modules} \simeq R/I\text{-cell-}R\text{-modules}.$$

Furthermore, it is proved in [2, 6.1] that the localizing subcategory generated by R/I is also generated by a Koszul complex $K(x_1, x_2, \ldots, x_n)$ where $I = (x_1, x_2, \ldots, x_n)$, so that we could equally have cellularized with respect to the Koszul complex.

6. Hasse equivalences

The idea here is that if a ring (or differential graded algebra) R is expressed as the pullback of a diagram of rings, the Cellularization Principle lets us build up the model category of differential graded R-modules from categories of modules over the terms. See also [6] for a more general treatment. We apply the standard context of Proposition 2.7 with $\mathbb{M}$ the category of R-modules.

6.A. **Diagrams of modules.** To describe $\mathbb{N}$ we start with a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\alpha"] \arrow[d, "\beta"'] & R^l \arrow[d, "\gamma"] \\ R^c \arrow[r, "\delta"'] & R^t. \end{tikzcd}$$

of rings.

Example 6.1. The classical Hasse principle is built on the pullback square

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\ \prod_p \mathbb{Z}_p^{\wedge} \arrow[r] & (\prod_p \mathbb{Z}_p^{\wedge}) \otimes \mathbb{Q} \end{tikzcd}$$

Returning to the general case, we delete R and consider the diagram

$$R^{\lrcorner} = \left(\begin{tikzcd} & R^l \arrow[d, "g"] \\ R^c \arrow[r, "d"'] & R^t \end{tikzcd} \right)$$

with three objects. We may form the category $\mathbb{N} = R^{\lrcorner}$-mod of diagrams

$$\begin{tikzcd} & M^l \arrow[d, "h"] \\ M^c \arrow[r, "e"'] & M^t \end{tikzcd}$$

where M^l is an R^l-module, M^c is an R^c-module, M^t is an R^t-module and the maps h and e are module maps over the corresponding maps of rings. That is, $h : M^l \to g^* M^t$ is a map of R^l-modules and $e : M^c \to d^* M^t$ is a map of R^c-modules. We will return to model structures below. Since $R^{\lrcorner}$ is a diagram of R-algebras, termwise tensor product gives a functor

$$R^{\lrcorner} \otimes_R : R\text{-mod} \longrightarrow R^{\lrcorner}\text{-mod}.$$

Similarly, since R maps to the pullback $PR^{\lrcorner}$, pullback gives a functor

$$P : R^{\lrcorner}\text{-mod} \longrightarrow R\text{-mod}.$$

It is easily verified that these give an adjoint pair

$$R^{\lrcorner} \otimes_R : R\text{-mod} \rightleftarrows R^{\lrcorner}\text{-mod} : P\ .$$

We may then consider the unit

$$\eta : M \longrightarrow P(R^{\lrcorner} \otimes_R M).$$

Since R is the generator of the category of R-modules, we want to require that η is an equivalence when $M = R$, which is to say the original diagram of rings is a homotopy pullback.

On the other hand, we cannot expect the counit of the adjunction to be an equivalence since we can add any module to M^t without changing $PM^{\lrcorner}$. This is where the Cellularization Principle comes in. We should use the image of R to cellularize the category of diagrams of modules. Before we do this we should describe the model structure.

6.B. **Model structures.** We give categories of (differential graded) modules over a ring the (algebraically) projective model structure, with homology isomorphisms as weak equivalences and fibrations the surjections. The cofibrations are retracts of relative cell complexes, where the spheres are shifted copies of R. The category $R^{\lrcorner}$-mod gets the diagram-injective model structure in which cofibrations and weak equivalences are maps which have this property objectwise; the fibrant objects have γ and δ surjective. This diagram-injective model structure is shown to exist for more general diagrams of ring spectra in an appendix of the original versions of [4], see also [6], and the same proof works for DGAs.

Since extension of scalars is a left Quillen functor for the (algebraically) projective model structure for any map of DGAs, $R^{\lrcorner} \otimes_R -$ also preserves cofibrations and weak equivalences and is therefore also a left Quillen functor. We then apply the Cellularization Principle to obtain the following result.

Proposition 6.2. *Assume given a commutative square of DGAs which is a homotopy pullback. The adjunction induces a Quillen equivalence*

$$R\text{-}mod \xrightarrow{\simeq} R^{\lrcorner}\text{-}cell\text{-}R^{\lrcorner}\text{-}mod,$$

where cellularization is with respect to the image, $R^{\lrcorner}$, of the generating R-module R.

Proof: We apply Proposition 2.7, which states that if we cellularize the model categories with respect to corresponding sets of objects, we obtain a Quillen adjunction.

In the present case, we cellularize with respect to the single R-module R on the left, and the corresponding diagram $R^{\lrcorner}$ on the right. Since the original diagram of rings is a homotopy pullback, the unit of the adjunction is an equivalence for R, and we see that the generator R and the generator $R^{\lrcorner}$ correspond under the equivalence, as required in the hypothesis in Part (2) of Proposition 2.7.

Since R is cofibrant and generates R-mod, cellularization with respect to R has no effect on R-mod and we obtain the stated equivalence with the cellularization of $R^{\lrcorner}$-mod with respect to the diagram $R^{\lrcorner}$. □

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